

# Generalized Fisher information matrix in nonextensive systems with spatial correlation

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By using the  $q$ -Gaussian distribution derived by the maximum entropy method for spatially correlated  $N$ -unit nonextensive systems, we have calculated the generalized Fisher information matrix of  $g_{\theta_n, \theta_m}$  for  $(\theta_1, \theta_2, \theta_3) = (\mu_q, \sigma_q^2, s)$ , where  $\mu_q$ ,  $\sigma_q^2$ , and  $s$  denote the mean, variance, and degree of spatial correlation, respectively, for a given entropic index  $q$ . It has been shown from the Cramér-Rao theorem that (1) an accuracy of an unbiased estimate of  $\mu_q$  is improved (degraded) by a negative (positive) correlation  $s$ , (2) that of  $\sigma_q^2$  is worsened with increasing  $s$ , and (3) that of  $s$  is much improved for  $s \approx -1/(N-1)$  or  $s \approx 1.0$  though it is worst at  $s = (N-2)/2(N-1)$ . Our calculation provides a clear insight to the long-standing controversy whether the spatial correlation is beneficial or detrimental to decoding in neuronal ensembles. We discuss also a calculation of the  $q$ -Gaussian distribution applying the superstatistics to the Langevin model subjected to spatially correlated inputs.

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## I. INTRODUCTION

It is well known that the Fisher information plays an important role in statistical mechanics and information theory (for review see [1]). The Fisher information is a useful tool in evaluating an accuracy of information decoding providing the lower bound for estimation errors of unbiased estimates in the Cramér-Rao theorem [1]. The Fisher information expresses the metric tensor in the Riemannian space spanned by the probability distribution functions (PDFs) in the information geometry [2]. Calculations of the Fisher information have been made for various systems such as neuronal ensembles [3–17]. Neurons in ensembles communicate information, emitting short voltage pulses called spikes, which propagate through axons and dendrites to neurons in the next stage (for review see [18–22] and related references therein). Main issues on the neuronal code are whether the information is encoded in the rate of firings (*rate code*) or in the firing times (*temporal code*), and whether the information is encoded in the activity of a single (or very few) neuron or that of a large number of neurons (*population code*). A recent success in brain-machine interface [23] suggests that the population code for the firing rate is employed in sensory and motor neurons, although it is still unclear what kinds of codes are adopted in higher-level cortical neurons.

The theoretical study of the Fisher information has been performed for a discussion on the accuracy of decoding and the efficiency of information transmission [3–17]. Calculations of the Fisher information have been made mainly for uncorrelated (independent) systems because of a mathematical simplicity. It has been shown that in independent systems, the Fisher information increases proportionally to the ensemble size [4, 8, 10, 11]. However, the correlation among constituent elements is inevitable in real systems. In neuronal ensembles, for example, statistical dependence among consisting neurons would be expected because each neuron may receive the same external inputs and because consisting neu-

rons are generally interconnected [18–22]. There has been a long-standing controversy how correlation affects the efficiency of population coding. Some researchers have shown that the correlation enhances the effectiveness of neural population code [9, 12], while some have claimed that the correlation hinders the population code [5–8, 10, 11]. In particular, the Fisher information is shown to saturate to a finite value as the system size grows in the presence of a positive correlation [8, 10, 11]. This raises questions on the role of correlation in information decoding.

In the last decade, much attention has been paid to the nonextensive statistics since Tsallis proposed the so-called Tsallis entropy  $S_q$ . For  $N$ -unit systems,  $S_q$  is given by [24–27]

$$S_q = \frac{k_B}{q-1} \left[ 1 - \int p(\mathbf{x})^q d\mathbf{x} \right], \quad (1)$$

where  $q$  is the entropic index,  $k_B$  the Boltzmann constant,  $\mathbf{x} = \{x_i\}$  ( $i=1$  to  $N$ ),  $d\mathbf{x} = \prod_{i=1}^N dx_i$ , and  $p(\mathbf{x})$  denotes the multivariate PDF. In the limit of  $q \rightarrow 1.0$ , the Tsallis entropy given by Eq. (1) reduces to the Boltzmann-Gibbs-Shannon entropy,

$$S_1 = -k_B \int p(\mathbf{x}) \ln p(\mathbf{x}) d\mathbf{x}. \quad (2)$$

The Tsallis entropy is nonadditive because for  $p(A \cup B) = p(A)p(B)$ , we obtain

$$S_q(A \cup B) = S_q(A) + S_q(B) - \frac{(q-1)}{k_B} S_q(A) S_q(B). \quad (3)$$

The Tsallis entropy is superextensive, extensive, and subextensive for  $q < 1$ ,  $q = 1$ , and  $q > 1$ , respectively, and  $q-1$  expresses the degree of the nonextensivity. The PDF of  $p(\mathbf{x})$  in Eq. (1) is obtained by using the maximum entropy method (MEM) for the Tsallis entropy with some constraints. There are four possible MEMs at the moment: original method [24], un-normalized method [28], normalized method [25], and the optimal Lagrange multiplier (OLM) method [29]. The four methods are equivalent in the sense that distributions derived in them are easily transformed to each other

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[30]. A comparison among the four MEMs is made in Ref. [27]. The Tsallis entropy is a basis of the nonextensive statistics, which has been successfully applied to a wide class of systems with the long-range interaction and/or nonequilibrium (quasiequilibrium) states [26,27,31].

One of the alternative approaches to the nonextensive statistics besides the MEM is the superstatistics [32–34] (for a recent review, see [35]). In the superstatistics, it is assumed that *locally* the equilibrium state of a given system is described by the Boltzmann-Gibbs statistics and its global properties may be expressed by a superposition over the fluctuating intensive parameter (i.e., the inverse temperature) [32–35]. The superstatistics has been adopted in many kinds of subjects such as hydrodynamic turbulence [36–38], cosmic ray [39], and solar flares [40].

The generalized Fisher information (GFI) in the nonextensive statistics is defined by [41–48]

$$g_{\theta_n \theta_m} = qE \left[ \left( \frac{\partial \ln p(\mathbf{x})}{\partial \theta_n} \right) \left( \frac{\partial \ln p(\mathbf{x})}{\partial \theta_m} \right) \right], \quad (4)$$

where  $E[\cdot]$  stands for the expectation value over the PDF of  $p(\mathbf{x})$  [ $=p(\mathbf{x}|\theta)$ ], and  $\theta$  parameters specifying the PDF. Equation (4) is derived from the generalized Kullback-Leibler divergence which is in conformity with the Tsallis entropy [41–48]. In the limit of  $q \rightarrow 1.0$ , the GFI given by Eq. (4) reduces to the conventional one. In a previous paper [49], we discussed the effect of the spatial correlation on the Tsallis entropy and the GFI, calculating  $S_q$  and  $g_{\theta\theta}$  for  $\theta = \mu_q$ , where  $\mu_q$  stands for mean value [Eq. (6)]. It is the purpose of the present paper to extend the calculation to the GFI matrix of  $g_{\theta_n \theta_m}$  for  $(\theta_1, \theta_2, \theta_3) = (\mu_q, \sigma_q^2, s)$ , where  $\sigma_q^2$  and  $s$  express variance and degree of the spatial correlation, respectively [Eqs. (7) and (8)]. We will investigate the dependence of the GFI on  $s$ ,  $N$ , and  $q$  by using the PDF derived by the OLM-MEM [29]. Such detailed calculations of the GFI matrix have not been reported even for the extensive system ( $q=1.0$ ), as far as the author is aware of. The calculated GFI is expected to provide us with a clear insight to the controversy on a role of the spatial correlation discussed above. Quite recently, we have pointed out the possibility that input information to neuronal ensembles may be carried not only by mean but also by variance and/or correlation in firing rate within the population code hypothesis [50,51]. The inverse of the calculated GFI matrix expresses an accuracy of decoding when input information is carried by such population codes.

The paper is organized as follows. In Sec. II, we obtain the PDF by the OLM-MEM for spatially correlated nonextensive systems. In Sec. III, the maximum likelihood estimator for the inference of the parameters is discussed. In Sec. IV, analytic expressions for elements of the GFI matrix are presented with some model calculations. In Sec. V, the PDF for the Langevin model with spatially correlated inputs is calculated within the superstatistics [32,33], which is compared to that derived by the MEM in Sec. II. Section VI is devoted to conclusion with the relevance of our calculation to decoding in neuronal population code [50,51].

## II. MAXIMUM ENTROPY METHOD

We consider spatially correlated  $N$ -unit nonextensive systems for which the Tsallis entropy is given by Eq. (1) [24,25]. We derive the PDF,  $p(\mathbf{x})$ , by using the OLM-MEM [29] for the Tsallis entropy imposing the constraints given by [49]

$$1 = \int p(\mathbf{x}) d\mathbf{x}, \quad (5)$$

$$\mu_q = \frac{1}{N} \sum_i E_q[x_i], \quad (6)$$

$$\sigma_q^2 = \frac{1}{N} \sum_i E_q[(x_i - \mu_q)^2], \quad (7)$$

$$s\sigma_q^2 = \frac{1}{N(N-1)} \sum_i \sum_{j(\neq i)} E_q[(x_i - \mu_q)(x_j - \mu_q)], \quad (8)$$

where  $E_q[\cdot]$  denotes an expectation value averaged over the escort distribution function of  $P_q(\mathbf{x})$ ,

$$P_q(\mathbf{x}) = \frac{p(\mathbf{x})^q}{\int p(\mathbf{x})^q d\mathbf{x}}. \quad (9)$$

The OLM-MEM with the constraints given by Eqs. (5)–(8) leads to the PDF given by (for details, see Appendix B of Ref. [49])

$$p(\mathbf{x}) = \frac{1}{Z_q} \exp_q \left[ - \left( \frac{1}{2\nu_q \sigma_q^2} \right) \sum_{i=1}^N \sum_{j=1}^N C_{ij} (x_i - \mu_q)(x_j - \mu_q) \right], \quad (10)$$

with

$$Z_q = \begin{cases} r_s \left( \frac{2\nu_q \sigma_q^2}{q-1} \right)^{N/2} \prod_{i=1}^N B \left( \frac{1}{2}, \frac{1}{q-1} - \frac{i}{2} \right) & \text{for } 1 < q < 3, \\ r_s (2\pi \sigma_q^2)^{N/2} & \text{for } q = 1, \\ r_s \left( \frac{2\nu_q \sigma_q^2}{1-q} \right)^{N/2} \prod_{i=1}^N B \left( \frac{1}{2}, \frac{1}{1-q} + \frac{(i+1)}{2} \right) & \text{for } q < 1, \end{cases} \quad (11)$$

$$C_{ij} = c_0 \delta_{ij} + c_1 (1 - \delta_{ij}), \quad (12)$$

$$c_0 = \frac{[1 + (N-2)s]}{(1-s)[1 + (N-1)s]}, \quad (13)$$

$$c_1 = - \frac{s}{(1-s)[1 + (N-1)s]}, \quad (14)$$

$$r_s = \{(1-s)^{N-1} [1 + (N-1)s]\}^{1/2}, \quad (15)$$

$$\nu_q = \frac{[(N+2) - Nq]}{2}, \tag{16}$$

where  $B(p, q)$  denotes the beta function and  $\exp_q(x)$  the  $q$ -exponential function defined by

$$\exp_q(x) = [1 + (1 - q)x]_+^{1/(1-q)}, \tag{17}$$

with  $[x]_+ = \max(x, 0)$ . We hereafter assume that the entropic index  $q$  takes a value,

$$0 < q < 1 + \frac{2}{N} \leq 3, \tag{18}$$

because  $p(x)$  given by Eq. (10) has the probability properties with  $\nu_q > 0$  for  $q < 1 + 2/N$  and because the Tsallis entropy is stable for  $q > 0$  [52].

In the limit of  $q=1.0$ , the PDF given by Eq. (10) becomes the multivariate Gaussian distribution given by

$$p(\mathbf{x}) = \frac{1}{Z_1} \exp \left[ - \left( \frac{1}{2\sigma_1^2} \right) \sum_{ij} C_{ij} (x_i - \mu_1)(x_j - \mu_1) \right]. \tag{19}$$

### III. MAXIMUM LIKELIHOOD ESTIMATOR

The logarithmic likelihood estimator for  $M$  sets of data of  $\mathbf{x}_m = \{x_{im}\}$  ( $i=1$  to  $N$ ,  $m=1$  to  $M$ ) is given by

$$\ln L(\theta) = \sum_{m=1}^M \ln p(\mathbf{x}_m | \theta) = - \left( \frac{1}{q-1} \right) \sum_{m=1}^M \ln U(\mathbf{x}_m) - M \ln Z_q, \tag{20}$$

with

$$U(\mathbf{x}_m) = 1 + \frac{(q-1)}{2\nu_q\sigma_q^2} \sum_{ij} C_{ij} (x_{im} - \mu_q)(x_{jm} - \mu_q). \tag{21}$$

Variational conditions for parameters of  $\theta = \mu_q, \sigma_q^2$ , and  $s$  lead to

$$\frac{\partial \ln L}{\partial \mu_q} = \frac{1}{\nu_q\sigma_q^2} \sum_{m=1}^M \sum_{ij} \frac{C_{ij}(x_{im} - \mu_q)}{U(\mathbf{x}_m)} = 0, \tag{22}$$

$$\frac{\partial \ln L}{\partial \sigma_q^2} = \frac{1}{2\nu_q\sigma_q^4} \sum_{m=1}^M \sum_{ij} \frac{C_{ij}(x_{im} - \mu_q)(x_{jm} - \mu_q)}{U(\mathbf{x}_m)} - \frac{MN}{2\sigma_q^2} = 0, \tag{23}$$

$$\begin{aligned} \frac{\partial \ln L}{\partial s} = & - \frac{1}{2\nu_q\sigma_q^2} \sum_{m=1}^M \sum_{ij} \frac{(dC_{ij}/ds)(x_{im} - \mu_q)(x_{jm} - \mu_q)}{U(\mathbf{x}_m)} \\ & + \frac{MN(N-1)}{2(1-s)[1+(N-1)s]} = 0. \end{aligned} \tag{24}$$

After some calculations using Eqs. (12)–(14) and Eqs. (22)–(24), we obtain

$$\mu_q = \frac{\sum_m \sum_i x_{im} U(\mathbf{x}_m)^{-1}}{N \sum_m U(\mathbf{x}_m)^{-1}}, \tag{25}$$

$$\sigma_q^2 = \frac{1}{\nu_q MN} \sum_m \sum_i \frac{(x_{im} - \mu_q)^2}{U(\mathbf{x}_m)}, \tag{26}$$

$$s\sigma_q^2 = \frac{1}{\nu_q MN(N-1)} \sum_m \sum_i \sum_{j(i \neq j)} \frac{(x_{im} - \mu_q)(x_{jm} - \mu_q)}{U(\mathbf{x}_m)}, \tag{27}$$

from which  $\mu_q, \sigma_q^2$ , and  $s$  are self-consistently determined.

In the case of  $q=1.0$ , Eqs. (25)–(27) become

$$\mu_1 = \frac{1}{MN} \sum_m \sum_i x_{im}, \tag{28}$$

$$\sigma_1^2 = \frac{1}{MN} \sum_m \sum_i (x_{im} - \mu_1)^2, \tag{29}$$

$$s\sigma_1^2 = \frac{1}{MN(N-1)} \sum_m \sum_i \sum_{j(i \neq j)} (x_{im} - \mu_1)(x_{jm} - \mu_1). \tag{30}$$

### IV. GENERALIZED FISHER INFORMATION

We have calculated elements of the GFI matrix given by Eq. (4) with a basis of  $(\theta_1, \theta_2, \theta_3) = (\mu_q, \sigma_q^2, s)$ , as given by (for details, see the Appendix)

$$\mathbf{G} = \begin{pmatrix} \frac{N}{\sigma_q^2[1+(N-1)s]} & 0 & 0 \\ 0 & \frac{N\nu_q}{2\sigma_q^4} & -\frac{N(N-1)\nu_qs}{2\sigma_q^2(1-s)[1+(N-1)s]} \\ 0 & -\frac{N(N-1)\nu_qs}{2\sigma_q^2(1-s)[1+(N-1)s]} & \frac{N(N-1)[1+(N-1)\nu_qs^2]}{2(1-s)^2[1+(N-1)s]^2} \end{pmatrix}. \tag{31}$$

The positive definiteness of  $g_{\theta\theta}$  in Eq. (31) imposes the condition on conceivable values of  $s$  and  $q$  given by

$$-\frac{1}{(N-1)} \equiv s_L < s \leq 1, \quad (32)$$

$$q \leq 1 + \frac{2}{N}. \quad (33)$$

The physical origin of Eq. (32) is expressed by (see Appendix C in Ref. [49])

$$0 \leq E_q[(X - \mu_q)^2] \leq \frac{1}{N} \sum_i E_q[(x_i - \mu_q)^2] = \sigma_q^2, \quad (34)$$

which signifies that the global fluctuation in  $X (=N^{-1}\sum_i x_i)$  is smaller than the average of local fluctuations in  $\{x_i\}$ . The condition given by Eq. (33) is satisfied by  $q$  in Eq. (18).

In the limit of  $q=1.0$  where  $\nu_q=1.0$ , Eq. (31) reduces to

$$\mathbf{G} = \begin{pmatrix} \frac{N}{\sigma_1^2[1+(N-1)s]} & 0 & 0 \\ 0 & \frac{N}{2\sigma_1^4} & -\frac{N(N-1)s}{2\sigma_1^2(1-s)[1+(N-1)s]} \\ 0 & -\frac{N(N-1)s}{2\sigma_1^2(1-s)[1+(N-1)s]} & \frac{N(N-1)[1+(N-1)s^2]}{2(1-s)^2[1+(N-1)s]^2} \end{pmatrix}, \quad (35)$$

which is in agreement with the result obtained directly from the multivariate Gaussian distribution given by Eq. (19).

In the limit of  $s=0.0$  (i.e., no correlation), the GFI matrix given by Eq. (31) becomes

$$\mathbf{G} = \begin{pmatrix} \frac{N}{\sigma_q^2} & 0 & 0 \\ 0 & \frac{N\nu_q}{2\sigma_q^4} & 0 \\ 0 & 0 & \frac{N(N-1)}{2} \end{pmatrix}, \quad (36)$$

whose elements of  $g_{\mu_q\mu_q}$  and  $g_{\sigma_q^2\sigma_q^2}$  agree with those obtained previously in Ref. [48].

The Cramér-Rao theorem implies that the lower bound of an unbiased estimate of the parameters is expressed by the inverse of the GFI matrix, which is given by

$$\mathbf{G}^{-1} = \begin{pmatrix} \frac{\sigma_q^2[1+(N-1)s]}{N} & 0 & 0 \\ 0 & \frac{2\sigma_q^4[1+(N-1)\nu_qs^2]}{N\nu_q} & \frac{2\sigma_q^2s(1-s)[1+(N-1)s]}{N} \\ 0 & \frac{2\sigma_q^2s(1-s)[1+(N-1)s]}{N} & \frac{2(1-s)^2[1+(N-1)s]^2}{N(N-1)} \end{pmatrix}. \quad (37)$$

Equations (31) and (37) are the main result of our study. In what follows, we examine the  $s$ ,  $N$ , and  $q$  dependences of the inversed GFI matrix of  $h_{\theta\theta} \equiv (\mathbf{G}^{-1})_{\theta\theta}$  with some model calculations which are presented in Figs. 1–3.

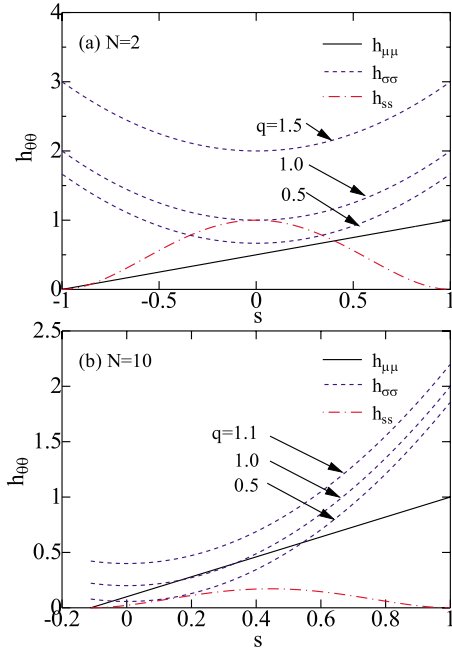


FIG. 1. (Color online) The  $s$  dependence of inverses of the GFI,  $h_{\mu_q\mu_q}$  (solid curves),  $h_{\sigma_q^2\sigma_q^2}$  (dashed curves), and  $h_{ss}$  (chain curves), with (a)  $N=2$  and (b)  $N=10$  for various  $q$  ( $\mu_q=0.0$  and  $\sigma_q^2=1.0$ ).

### A. $s$ dependence

Equation (37) shows (1)  $h_{\mu_q\mu_q}=0.0$  at  $s=s_L$ , (2)  $h_{\sigma_q^2\sigma_q^2}$  has a minimum at  $s=0.0$ , and (3)  $h_{ss}$  vanishes at  $s=s_L$  and  $s=1.0$ . The maximum of  $h_{ss}$  locates at  $s=(N-2)/2(N-1) \equiv s_M$ , which becomes  $s_M=0.5$  for a large  $N$ . Figure 1(a) shows the  $s$  dependence of the inverse of the GFI for  $N=2$  which is expressed by

$$\mathbf{G}^{-1} = \begin{pmatrix} \frac{\sigma_q^2(1+s)}{2} & 0 & 0 \\ 0 & \frac{\sigma_q^4(1+\nu_q s^2)}{\nu_q} & \sigma_q^2 s(1-s^2) \\ 0 & \sigma_q^2 s(1-s^2) & (1-s^2)^2 \end{pmatrix}. \quad (38)$$

With increasing  $s$  from  $s=s_L=-1.0$ ,  $h_{\mu_q\mu_q}$  is linearly increased.  $h_{ss}$  and  $h_{\sigma_q^2\sigma_q^2}$  are symmetric with respect to  $s=0.0$

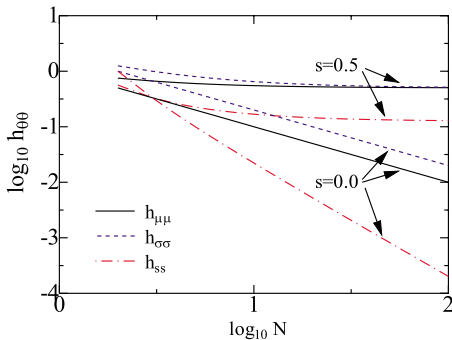


FIG. 2. (Color online) The  $N$  dependence of inverses of the GFI,  $h_{\mu_q\mu_q}$  (solid curves),  $h_{\sigma_q^2\sigma_q^2}$  (dashed curves), and  $h_{ss}$  (chain curves), for  $s=0.0$  and  $s=0.5$  ( $q=1.0$ ,  $\mu_q=0.0$ , and  $\sigma_q^2=1.0$ ).

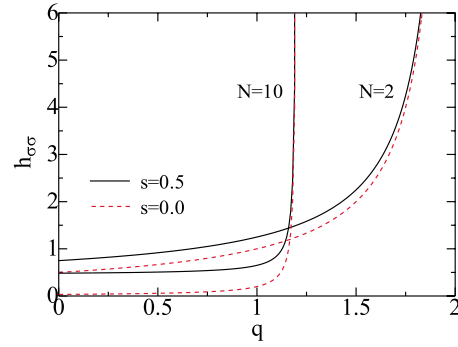


FIG. 3. (Color online) The  $q$  dependence of inverses of the GFI,  $h_{\sigma_q^2\sigma_q^2}$ , for  $s=0.0$  (dashed curves) and  $s=0.5$  (solid curves) with  $N=2$  and  $N=10$  ( $\mu_q=0.0$  and  $\sigma_q^2=1.0$ ).

where  $h_{ss}$  ( $h_{\sigma_q^2\sigma_q^2}$ ) has a maximum (minimum). Figure 1(b) shows a similar plot for  $N=10$  for which  $s_L=-0.11$ . With increasing  $s$  from  $s=-0.11$ ,  $h_{\mu_q\mu_q}$  is linearly increased.  $h_{ss}$  has a maximum at  $s=s_M=0.44$  and vanishes at  $s=-0.11$  and  $s=1.0$ .

### B. $N$ dependence

We note in Eq. (37) that for  $s=0$ , the GFI is proportional to  $N$ . For a finite positive  $s$ , however, they show the saturation when  $N$  is increased: for  $N \rightarrow \infty$ , we obtain  $h_{\mu_q\mu_q} = \sigma_q^2 s$ ,  $h_{\sigma_q^2\sigma_q^2} = 2\sigma_q^4 s^2$ , and  $h_{ss} = 2s^2(1-s)^2$ . For a negative  $s$ , inverse matrix elements tend to vanish as  $N$  approaches  $(1+|s|)/|s|$ . The calculated  $N$  dependence of  $h_{\theta\theta}$  is plotted in Fig. 2, where inversed matrix elements for  $s=0.5$  saturate at  $N \geq 10$ , although those for  $s=0.0$  are proportional to  $N^{-1}$ .

### C. $q$ dependence

Equation (37) shows that  $h_{\mu_q\mu_q}$  and  $h_{ss}$  are independent of  $q$ , while  $h_{\sigma_q^2\sigma_q^2}$  is increased with increasing  $q$  from  $q=0$ . This increase is due to a factor of  $\nu_q$  in Eq. (16), which is decreased with increasing  $q$  and which diverges at  $1+2/N$ : note that  $\nu_q = N/2 + 1$ ,  $1.0$ , and  $0.0$  for  $q=0.0$ ,  $q=1.0$ , and  $q=1+2/N$ , respectively. The calculated  $q$  dependence of  $h_{\sigma_q^2\sigma_q^2}$  is plotted in Fig. 3, where it diverges at  $q=2.0$  ( $q=1.2$ ) for  $N=2$  ( $N=10$ ).

## V. DISCUSSION

We have discussed the GFI for the  $q$ -Gaussian distribution derived by the MEM [24,25,29]. It is possible to derive the  $q$ -Gaussian distribution by using the Langevin model within the superstatistics [32,33]. We consider an ensemble consisting of  $N$  elements in a given system. The dynamics of a variable  $x_i$  ( $i=1$  to  $N$ ) is assumed to be described by the Langevin model given by

$$\frac{dx_i}{dt} = -\lambda x_i + I_i(t). \quad (39)$$

Here  $\lambda$  denotes the relaxation rate and input signals  $I_i(t)$  have variability defined by

$$I_i(t) = I(t) + \delta I_i(t), \quad (40)$$

with

$$\langle \delta I_i(t) \rangle = 0, \quad (41)$$

$$\langle \delta I_i(t) \delta I_j(t') \rangle = 2D[\delta_{ij} + s_I(1 - \delta_{ij})]\delta(t - t'), \quad (42)$$

where the bracket  $\langle \cdot \rangle$  signifies the ensemble average, and  $2D$  and  $s_I$  denote the variance and degree of the spatial correlation, respectively. The variability in Eq. (42) arises from noise and/or heterogeneity in consisting elements. The origin of the spatial correlation may be common external inputs and/or couplings among elements.

The Fokker-Planck equation (FPE) for the PDF of  $\pi(\mathbf{x}, t)$  for  $\mathbf{x} = \{x_i\}$  is given by

$$\begin{aligned} \frac{\partial \pi(\mathbf{x}, t)}{\partial t} = & \sum_i \frac{\partial}{\partial x_i} [\lambda x_i - I(t)] \pi(\mathbf{x}, t) \\ & + D \sum_i \sum_j Q_{ij} \frac{\partial^2}{\partial x_i \partial x_j} \pi(\mathbf{x}, t), \end{aligned} \quad (43)$$

with the covariance matrix  $\mathbf{Q}$  whose elements are given by

$$Q_{ij} = \delta_{ij} + s_I(1 - \delta_{ij}). \quad (44)$$

The solution of FPE (43) is given by

$$\begin{aligned} \pi(\mathbf{x}, t) = & \left( \frac{1}{r_s [2\pi\sigma^2]^{N/2}} \right) \\ & \times \exp \left[ -\frac{1}{2\sigma^2} \sum_i \sum_j C_{ij} (x_i - \mu)(x_j - \mu) \right], \end{aligned} \quad (45)$$

where  $\mu$ ,  $\sigma^2$ , and  $s$  obey equations of motion given (argument  $t$  being suppressed) by

$$\frac{d\mu}{dt} = -\lambda\mu + I, \quad (46)$$

$$\frac{d\sigma^2}{dt} = -2\lambda\sigma^2 + 2D, \quad (47)$$

$$\frac{ds}{dt} = -\left(\frac{2D}{\sigma^2}\right)(s - s_I), \quad (48)$$

with  $C_{ij}$  and  $r_s$  being defined by Eqs. (12) and (15), respectively. We note in Eqs. (46)–(48) that  $\mu(t)$  is decoupled from  $\sigma^2(t)$  and  $s(t)$ , and that  $\sigma^2(t)$  is independent of  $s(t)$  although  $s(t)$  depends on  $\sigma^2(t)$ . In the stationary state, we obtain

$$\mu = I/\lambda, \quad \sigma^2 = \frac{D}{\lambda}, \quad s = s_I. \quad (49)$$

After the concept in the superstatistics [32–35], we assume that a model parameter of  $\beta$  ( $=1/\sigma^2 = \lambda/D$ ) fluctuates, and that its distribution is expressed by the  $\chi^2$  distribution with rank  $n$ ,

$$f(\beta) = \frac{1}{\Gamma(n/2)} \left( \frac{n}{2\beta_0} \right)^{n/2} \beta^{n/2-1} e^{-n\beta/2\beta_0}, \quad (n = 1, 2, \dots), \quad (50)$$

where  $\Gamma(x)$  is the gamma function. Average and variance of  $\beta$  are given by  $\langle \beta \rangle_{\beta=\beta_0}$  and  $(\langle \beta^2 \rangle_{\beta=\beta_0} - \beta_0^2)/\beta_0^2 = 2/n$ , respectively. Taking the average of  $\pi(\mathbf{x})$  over  $f(\beta)$ , we obtain the stationary distribution given by

$$p(\mathbf{x}) = \int_0^\infty \pi(\mathbf{x}) f(\beta) d\beta, \quad (51)$$

$$= \frac{1}{Z_q} \exp_q \left[ -\frac{1}{2\gamma_q} \sum_i \sum_j C_{ij} (x_i - \mu)(x_j - \mu) \right], \quad (52)$$

with

$$Z_q = \begin{cases} r_s \left( \frac{2\gamma_q}{q-1} \right)^{N/2} \prod_{i=1}^N B\left(\frac{1}{2}, \frac{1}{q-1} - \frac{i}{2}\right) & \text{for } q > 1, \\ r_s (2\pi\gamma_q)^{N/2} & \text{for } q = 1, \end{cases} \quad (53)$$

$$q = 1 + \frac{2}{(N+n)}, \quad (54)$$

$$\gamma_q = \frac{n}{\beta_0(N+n)} = \frac{(N+2) - Nq}{2\beta_0}, \quad (55)$$

where  $r_s$  is given by Eq. (15). In the limit of  $n \rightarrow \infty$  ( $q \rightarrow 1.0$ ) where  $f(\beta) \rightarrow \delta(\beta - \beta_0)$ , the PDF reduces to the multivariate Gaussian distribution given by

$$p(\mathbf{x}) = \frac{1}{Z_1} \exp \left[ -\frac{\beta_0}{2} \sum_i \sum_j C_{ij} (x_i - \mu)(x_j - \mu) \right], \quad (56)$$

which agrees with Eq. (45) for  $\beta_0 = \lambda/D = 1/\sigma^2$ .

We note that the PDF given by Eq. (52) is equivalent to that given by Eq. (10) derived by the MEM when we read  $\mu = \mu_q$  and  $\gamma_q = \nu_q \sigma_q^2$ , besides the fact that the former is defined for  $1 \leq q \leq [1 + 2/(N+n)] < 2$  [Eq. (54)] while the latter for  $0 < q < (1 + 2/N) < 3$  [Eq. (18)].

In the limit of  $s=0$  (i.e., no spatial correlation), Eq. (52) reduces to

$$p(\mathbf{x}) \propto \exp_q \left[ -\frac{1}{2\gamma_q} \sum_i (x_i - \mu)^2 \right], \quad (57)$$

$$\propto p(x_1) \otimes_q p(x_2) \otimes_q \cdots \otimes_q p(x_N), \quad (58)$$

with

$$p(x_i) \propto \exp_q \left[ -\frac{1}{2\gamma_q} (x_i - \mu)^2 \right], \quad (59)$$

where the  $q$  product is defined by [53]

$$x \otimes_q y = [x^{1-q} + y^{1-q} - 1]^{1/(1-q)}. \quad (60)$$

Note that in deriving Eq. (58), the normalization factors of  $p(x_i)$  are not taken into account.

**VI. CONCLUSION**

We have calculated the GFI matrix in spatially correlated nonextensive systems. From the Cramér-Rao theorem, the calculated GFI implies the following: (i) an accuracy of an estimate of  $\mu_q$  is improved (degraded) by a negative (positive) correlation, (ii) that of  $\sigma_q^2$  is worsen with increasing  $s$ , (iii) that of  $s$  is much improved for  $s \approx -1/(N-1)$  and  $s \approx 1.0$  while it is worst at  $s = s_M = (N-2)/2(N-1)$ , (iv) those of all parameters are improved with increasing  $N$ , and (v) that of  $\sigma_q^2$  is worsen with increasing  $q$  at  $q > 1$  and its estimation is impossible for  $q \geq 1 + 2/N$ , while those of  $\mu_q$  and  $s$  are independent of  $q$ .

The points (i) and (iv) are consistent with previous results for extensive systems ( $q=1.0$ ) [7,8,10]. The point (iii) shows that if input information is carried by synchrony within the population code hypothesis [50,51], its decoding accuracy may be improved either by small or large correlation independently of  $q$  [the point (v)]. Our calculation concerns the long-standing controversy on a role of the synchrony in neuronal ensembles [5–12].

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**APPENDIX: CALCULATIONS OF THE GENERALIZED FISHER INFORMATION MATRIX**

First we express PDFs of  $p(\mathbf{x})$  in Eq. (10) in a compact form given by

$$p(\mathbf{x}) = \frac{U^{-b}}{Z_q}, \tag{A1}$$

with

$$U = 1 + a^2 \sum_i \sum_j C_{ij} (x_i - \mu_q)(x_j - \mu_q), \tag{A2}$$

$$Z_q = \begin{cases} \frac{r_s}{a^N} \prod_{i=1}^N B\left(\frac{1}{2}, b - \frac{i}{2}\right) & \text{for } 1 < q < 3, \\ r_s (2\pi\sigma_q^2)^{N/2} & \text{for } q = 1, \\ \frac{r_s}{a^N} \prod_{i=1}^N B\left(\frac{1}{2}, -b + \frac{(i+1)}{2}\right) & \text{for } q < 1, \end{cases} \tag{A3}$$

$$a = \left( \frac{|q-1|}{2\nu_q \sigma_q^2} \right)^{1/2}, \tag{A4}$$

$$b = \frac{1}{q-1}. \tag{A5}$$

By using the unitary transformation, Eq. (A2) is transformed to

$$U = 1 + a^2 \sum_i \lambda_i y_i^2, \tag{A6}$$

where  $\lambda_i$  and  $y_i$  express eigenvalues and eigenvectors, respectively. We obtain  $\lambda_i$  given by

$$\lambda_i = \frac{1}{[1 + (N-1)s]} \quad \text{for } i = 1, \tag{A7}$$

$$= \frac{1}{(1-s)} \quad \text{for } 1 < i \leq N. \tag{A8}$$

Explicit expressions for  $y_i$  are not necessary for our discussion, except for  $y_1$  given by

$$y_1 = \frac{1}{\sqrt{N}} \sum_i (x_i - \mu_q). \tag{A9}$$

Taking the derivatives of  $\ln p(\mathbf{x})$  with respect to parameters of  $\mu_q$ ,  $\sigma_q^2$ , and  $s$ , and performing tedious calculations with Eq. (4), we may obtain the GFI matrix elements given by Eq. (31). In deriving them, we have employed the following expectation values:

$$E\left[\frac{1}{U}\right] = \frac{(b-N/2)}{b}, \tag{A10}$$

$$E\left[\frac{y_i^2}{U}\right] = \frac{1}{2a^2 b \lambda_i}, \tag{A11}$$

$$E\left[\frac{y_i^2}{U^2}\right] = \frac{(b-N/2)}{2a^2 b (b+1) \lambda_i}, \tag{A12}$$

$$E\left[\frac{y_i^4}{U^2}\right] = \frac{3}{4a^4 b (b+1) \lambda_i^2}, \tag{A13}$$

$$E\left[\frac{y_i^2 y_j^2}{U^2}\right] = \frac{1}{4a^4 b (b+1) \lambda_i \lambda_j} \quad \text{for } i \neq j, \tag{A14}$$

where  $E[\cdot]$  denotes the average over  $p(\mathbf{x})$ .

[1] B. R. Frieden, *Physics from Fisher Information: A Unification* (Cambridge University Press, Cambridge, 1998).  
 [2] S. Amari and H. Nagaoka, *Methods of Information Geometry* (AMS/Oxford University Press, Providence/New York, 2000).

[3] M. A. Paradiso, *Biol. Cybern.* **58**, 35 (1988).  
 [4] H. Seung and H. Sompolinsky, *Proc. Natl. Acad. Sci. U.S.A.* **90**, 10749 (1993).  
 [5] T. Gawne and B. Richmond, *Neuroscience* **13**, 2758 (1993).

- [6] E. Zohary and W. Newsome, *Nature (London)* **370**, 140 (1994).
- [7] S. Panzeri, S. R. Schultz, A. Treves, and E. T. Rolls, *Proc. R. Soc. London, Ser. A* **266**, 1001 (1999).
- [8] L. F. Abbott and P. Dayan, *Neural Comput.* **11**, 91 (1999).
- [9] R. Jenison, *J. Comput. Neurosci.* **3**, 285 (2000).
- [10] H. Sompolinsky, H. Yoon, K. Kang, and M. Shamir, *Phys. Rev. E* **64**, 051904 (2001).
- [11] M. Shamir and H. Sompolinsky, in *Advances in Neural Information Processing Systems*, edited by T. G. Dietterich, S. Becker, and Z. Ghahramani (MIT Press, Cambridge, MA, 2002), Vol. 14, pp. 277–284.
- [12] D. Johnson, *J. Comput. Neurosci.* **16**, 69 (2004).
- [13] S. Wu, S. Amari, and H. Nakamura, *Neural Comput.* **14**, 999 (2002).
- [14] N. S. Harper and D. McAlpine, *Nature (London)* **430**, 682 (2004).
- [15] S. D. Wilke and C. W. Eurich, *Neurocomputing* **44-46**, 1023 (2002).
- [16] S. Wu, S. Amari, and H. Nakamura, *Neural Networks* **17**, 205 (2004).
- [17] T. Toyozumi, K. Aihara, and S. Amari, *Phys. Rev. Lett.* **97**, 098102 (2006).
- [18] F. Rieke, D. Warland, R. Steveninck, and W. Bialek, *Spikes—Exploring the Neural Code* (MIT Press, Cambridge, 1996).
- [19] W. M. Usrey and R. C. Reid, *Annu. Rev. Physiol.* **61**, 435 (1999).
- [20] R. C. deCharms and A. Zador, *Annu. Rev. Neurosci.* **23**, 613 (2000).
- [21] B. B. Averbeck, P. E. Latham, and A. Pouget, *Nat. Rev. Neurosci.* **7**, 358 (2006).
- [22] H. Hasegawa, in *Neuronal Network Research Horizons*, edited by M. L. Weiss (Nova Science, New York, 2007), p. 61.
- [23] J. M. Carmena, M. A. Lebedev, R. E. Crist, J. E. O’Doherty, D. M. Santucci, D. F. Dimitrov, P. G. Patil, C. S. Henriquez, and M. A. L. Nicolelis, *PLoS Biol.* **1**, 193 (2003); R. A. Anderson, S. Musallam, and B. Pesaram, *Curr. Opin. Neurobiol.* **23**, 613 (2004).
- [24] C. Tsallis, *J. Stat. Phys.* **52**, 479 (1988).
- [25] C. Tsallis, R. S. Mendes, and A. R. Plastino, *Physica A* **261**, 534 (1998).
- [26] C. Tsallis, in *Nonextensive Statistical Mechanics and Its Application*, edited by S. Abe and Y. Okamoto (Springer-Verlag, Berlin, 2001), p. 3.
- [27] C. Tsallis, *Physica D* **193**, 3 (2004).
- [28] E. M. F. Curado and C. Tsallis, *J. Phys. A* **24**, L69 (1991); **25**, 1019 (1992).
- [29] S. Martinez, F. Nicolas, F. Pennini, and A. Plastino, *Physica A* **286**, 489 (2000).
- [30] G. L. Ferri, S. Martinez, and A. Plastino, *J. Stat. Mech.: Theory Exp.* (2005) P04009.
- [31] Lists of many applications of the nonextensive statistics are available at <http://tsallis.cat.cbpf.br/biblio.htm>
- [32] G. Wilk and Z. Włodarczyk, *Phys. Rev. Lett.* **84**, 2770 (2000).
- [33] C. Beck, *Phys. Rev. Lett.* **87**, 180601 (2001).
- [34] C. Beck and E. G. D. Cohen, *Physica A* **322**, 267 (2003).
- [35] C. Beck, in *Anomalous Transport: Foundations and Applications*, edited by G. Radons *et al.* (Wiley, New York, 2007).
- [36] C. Beck, *Europhys. Lett.* **64**, 151 (2003).
- [37] A. M. Reynolds, *Phys. Rev. Lett.* **91**, 084503 (2003).
- [38] C. Beck, *Phys. Rev. Lett.* **98**, 064502 (2007).
- [39] C. Beck, *Physica A* **331**, 173 (2004).
- [40] M. Baiesi, M. Paczuski, and A. L. Stella, *Phys. Rev. Lett.* **96**, 051103 (2006).
- [41] S. Abe, *Phys. Rev. E* **68**, 031101 (2003).
- [42] P. W. Lambert, M. T. Martin, A. Plastino, and O. A. Rosso, *Physica A* **334**, 119 (2004).
- [43] J. Naudts, *J. Ineq. Pure Appl. Math.* **5**, 102 (2004).
- [44] F. Pennini and A. Plastino, *Physica A* **334**, 132 (2004).
- [45] M. Portesi, A. Plastino, and F. Pennini, *Physica A* **365**, 173 (2006).
- [46] M. Portesi, F. Pennini, and A. Plastino, *Physica A* **373**, 273 (2007).
- [47] M. Masi, e-print arXiv:cond-mat/0611300.
- [48] H. Hasegawa, *Phys. Rev. E* **77**, 031133 (2008).
- [49] H. Hasegawa, *Phys. Rev. E* **78**, 021141 (2008).
- [50] H. Hasegawa, *Phys. Rev. E* **78**, 031110 (2008).
- [51] H. Hasegawa, *Physica A* **388**, 499 (2009).
- [52] S. Abe, *Phys. Rev. E* **66**, 046134 (2002).
- [53] E. P. Borges, *Physica A* **340**, 95 (2004).